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R^4 couplings, the fundamental membrane and exceptional theta correspondences

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Abstract: This letter is an attempt to carry out a first-principle computation in M-theory using the point of view that the eleven-dimensional membrane gives the fundamental degrees of freedom of M-theory. Our aim is to derive the exact BPS R^4 couplings in M-theory compactified on a torus $\mathbb{T}^{d+1}$ from the toroidal BPS membrane, by pursuing the analogy with the one-loop string theory computation. We exhibit an $\mathrm{Sl}(3,\mathbb{Z})$ modular invariance hidden in the light-cone gauge (but obvious in the Polyakov approach), and recover the correct classical spectrum and membrane instantons; the summation measure however is incorrect. It is argued that the correct membrane amplitude should be given by an exceptional theta correspondence lifting $\mathrm{Sl}(3,\mathbb{Z})$ modular forms to $E_{d+1(d+1)}(\mathbb{Z})$ automorphic forms, generalizing the usual theta lift between $\mathrm{Sl}(2,\mathbb{Z})$ and $\mathrm{SO}(d,d,\mathbb{Z})$ in string theory. The exceptional correspondence $\mathrm{Sl}(3)\times E_{6(6)}\subset E_{8(8)}$ offers the interesting prospect of solving the membrane small volume divergence and unifying membranes with five-branes.

Keywords: M-Theory, String Duality.

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Introduction. Despite considerable insights brought about by the discovery of dualities, a tractable microscopic definition of non-perturbative string theory remains an outstanding problem hindering further progress. While several definitions are available, computational power is the issue: the eleven-dimensional membrane is strongly interacting, whereas the large-N limit of its M(atrix) theory cousins is still largely untamed. For particular supersymmetric situations however, one would expect all quantum fluctuations to cancel, leaving a hopefully tractable zero-mode problem. For example, the spectrum of BPS states in toroidal compactifications of M-theory has been reproduced from the BPS five-brane [1] or from M(atrix) theory (see e.g. [2] for a review along these lines). This amounts to a rather trivial check of the duality invariance of the proposed classical action. Somewhat less trivial would be to give a microscopic derivation of a particular amplitude, receiving contributions from supersymmetric states but still with a non-trivial summation measure. Such is the case of the eight-derivative R^4 couplings, exactly known in toroidal compactifications of M-theory on the basis of duality and supersymmetry [3]–[6]. In this work, we attempt a microscopic derivation of these couplings, contending that the eleven-dimensional membrane gives the fundamental degrees of freedom of M-theory. Due to our limited understanding of the quantization of the membrane, we proceed by analogy with the BPS string, and try to construct a partition function for a toroidal BPS membrane exhibiting at the same time world-volume modular invariance and target-space U-duality.[1] The classical membrane partition function reproduces the correct BPS spectrum and instanton saddle points, but is not U-duality invariant. We propose that the appropriate quantum modification is provided by exceptional theta correspondences from the theory of automorphic forms, in analogy with the symplectic theta correspondence arising in string theory. The explicit construction of such correspondences is outlined but left for future work. It is nevertheless expected to provide important clues about the quantization of the membrane, and possibly give a mechanism for the finiteness of membrane theory and unification of membranes and five-branes.

One-loop string amplitude. In order to motivate our approach, recall the computation of R^4 couplings in type-II string theory [3, 4]. The four-graviton scattering amplitude at eight derivative order receives contributions from tree-level, one-loop, together with an infinite series of D-instanton corrections,

$$f_{R^4} = \frac{2\zeta(3)V}{g_s^2 l_s^2} + f_{\text{1-loop}} + \mathcal{O}\left(e^{-1/g_s}\right) . \tag{1}$$

The tree-level term is essentially field-theoretical and depends only on the total volume V of the internal torus $\mathbb{T}^d$. The one-loop term on the other hand is a compli-

[1]Our goal bears some resemblance to the work [7], where a modular invariant partition function for the fivebrane is proposed; unfortunately their result is not invariant under target-space duality.

cated function of the torus moduli, invariant under target space duality $\mathrm{SO}(d,d,\mathbb{Z})$. It can be written as an integral over the fundamental domain of the Poincaré upper half-plane,

$$f_{\text{1-loop}} = 2\pi \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z_{d,d}\,, \qquad (\tau; g, B)\,, \tag{2}$$

where $Z_{d,d}$ is the partition function (or theta function) of the even self-dual lattice describing the toroidal compactification,

$$Z_{d,d}(\tau; g, B) = V \sum_{(m^i,n^i)\in\mathbb{Z}^{2d}} e^{-\frac{\pi}{\tau_2}(m^i+\tau n^i)g_{ij}(m^j+\bar\tau n^j)+2\pi i m^i B_{ij} n^j}\,. \tag{3}$$

Notice that only the zero-modes of the string coordinates contribute, all bosonic and fermionic oscillators having canceled in this supersymmetric amplitude. Their weight is given by the Polyakov action

$$S = \int d^2\sigma \sqrt{\gamma} g_{ij} \gamma^{ab} \partial_a X^i \partial_b X^j + 2\pi i \epsilon^{ab} B_{ij} \partial_a X^i \partial_b X^j\,, \tag{4}$$

evaluated on the classical (zero-mode) configuration and worldsheet unit-volume metric

$$X^i = m^i \sigma_1 + n^i \sigma_2\,, \qquad \gamma_{ab} = \frac{1}{\tau_2}\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}. \tag{5}$$

Sum over states. For the purpose of generalization to the eleven-dimensional membrane, it will be useful to understand this one-loop amplitude from several viewpoints. First, let us Poisson resum the windings m^i along the σ_1 direction into momenta m_i, and rewrite

$$f_{\text{1-loop}} = 2\pi \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{2-(d/2)}} \sum_{m_i,n^i} e^{-\pi\tau_2\left[(m_i+B_{ij}n^j)^2+(n^i)^2\right]-2\pi\tau_1 i m_i n^i}\,. \tag{6}$$

Ignoring for a moment the restriction $|\tau| > 1$ on the fundamental domain $\mathcal{F}$, we recognize τ_2 as the Schwinger parameter, while the integral over $\tau_1 \in [-1/2, 1/2]$ imposes the BPS constraint $m_i n^i = 0$. The one-loop amplitude can therefore be viewed as a *regulated* sum over the one-loop contributions of all perturbative half-BPS states, with internal momentum m_i, winding n^i and mass

$$\mathcal{M}^2 = \left(m_i + B_{ij} n^j\right)^2 + \left(n^i\right)^2. \tag{7}$$

In fact, one could have started from the light-cone description of the string to find the integrand in (6), discovered its hidden modular invariance under $\mathrm{Sl}(2,\mathbb{Z})$, and by hand restricted the integration to the fundamental domain, to render the integral UV finite. The Polyakov description produces this result automatically, and shows that all string theory amplitudes are UV finite.

Worldsheet instantons. Let us now revert to the "lagrangian" representation (3), and consider the large volume expansion (or equivalently the Fourier expansion in the periodic modulus B_{ij}) of the one-loop result. By the standard orbit decomposition method [8], the fundamental domain can be unfolded to produce

$$
\begin{aligned}
f_{\text{1-loop}} &= \frac{2\pi^2}{3} V + 2V \sum_{m^i \neq 0} \frac{1}{m^i g_{ij} m^j} + \\
&\quad + 4\pi V \sum_{m^{ij} \,/\, \mathrm{Sl}(2)} \mu\left(m^{ij}\right) \frac{\exp[-2\pi \sqrt{(m^{ij})^2} + 2\pi i B_{ij} m^{ij}]}{\sqrt{(m^{ij})^2}}
\end{aligned}
\tag{8}
$$

corresponding to the rank 0, 1 and 2 orbits of the integers (m^i, n^i) under the modular group $\mathrm{Sl}(2,\mathbb{Z})$, respectively. Again, the first and second term can be interpreted as the regulated contribution from the Kaluza-Klein states propagating in the loop, while the last term corresponds to worldsheet instantons wrapping subtori $\mathbb{T}^2 \subset \mathbb{T}^d$ with homology $m^{ij} = m^i n^j - m^j n^i$. Their classical action is recognized as the Nambu-Goto action, which follows from the fact that the Polyakov and Nambu-Goto action are equivalent upon imposing the equations of motion for the worldsheet metric γ_{ab}. More interestingly, the summation measure can be computed as the number theoretic function

$$
\mu\left(m^{ij}\right) = \sum_{n | m^{ij}} n
\tag{9}
$$

which is also fixed by requiring T-duality invariance. The measure (9) follows from the fact that the sum over the integers (m^i, n^i) is restricted to $\mathrm{Sl}(2,\mathbb{Z})$ orbits, which can be parameterized as

$$
\begin{pmatrix} m^i \\ n^i \end{pmatrix} = \begin{pmatrix} m & j & m^3 & m^4 & \cdots \\ 0 & n & n^3 & n^4 & \cdots \end{pmatrix}
\tag{10}
$$

with $m, n > 0$ and $0 \leq j < n$. The factor n in (9) arises from summing over $j = 0, \ldots, n-1$. Clearly, this factor would not be accessible from a Nambu-Goto description of the string.

Automorphic forms and theta lift. Finally, the one-loop result (8) can be encapsulated into a manifestly T-duality invariant result, using Eisenstein series of the T-duality group [6]. A particularly convenient representation is in terms of the vector representation,[2]

$$
f_{\text{1-loop}} = 2 \frac{\Gamma(d/2 - 1)}{\pi^{d/2-2}} \mathcal{E}^{\mathrm{SO}(d,d,\mathbb{Z})}_{\mathbf{V};\, s=d/2-1} \,,
\tag{11}
$$

where the divergence is regulated by analytic continuation to the complex s-plane, as in standard zeta function regularisation. This identification is in particular supported

[2]More precisely, the one-loop amplitude is equal to the *residue* of the Eisenstein series at the pole at $s = d/2 - 1$. We thank D. Kazhdan for pointing this out to us.

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by the identity [6]

$$\left[\Delta_{\mathrm{SO}(d,d)} - 2\Delta_{\mathrm{Sl}(2)} + \frac{d(d-2)}{4}\right] Z_{d,d} = 0 \tag{12}$$

satisfied by the BPS string partition function: after integrating by parts, this implies that the one-loop contribution is an eigenmode of the invariant laplacian on $\mathrm{SO}(d,d,\mathbb{R})/\mathrm{SO}(d)\times\mathrm{SO}(d)$ with eigenvalue $d(d-2)/4$, as is the case of the r.h.s. in (11). In fact, $Z_{d,d}$ and $f_{\text{1-loop}}$ are eigenmodes of all invariant differential operators on $\mathrm{Sl}(2)/\mathrm{U}(1)\times[\mathrm{SO}(d,d,\mathbb{R})/\mathrm{SO}(d)\times\mathrm{SO}(d)]$.

For higher point functions, the structure of the one-loop amplitude remains essentially as in (2), except for the insertion of a modular form $\Phi(\tau,\bar\tau)$ incorporating the effect of the oscillators and vertex insertions. The integrated amplitude is still T-duality invariant. From the mathematical point of view, this provides a "theta" lift of $\mathrm{Sl}(2,\mathbb{Z})$ modular forms to automorphic forms on $\mathrm{SO}(d,d,\mathbb{R})/\mathrm{SO}(d)\times\mathrm{SO}(d)$,

$$\Phi \longrightarrow \int_{\mathcal{F}(\mathrm{Sl}(2,\mathbb{Z}))} \frac{d\tau d\bar\tau}{\tau_2^2} Z_{d,d}(\tau;g,B)\ \Phi(\tau,\bar\tau)\,, \tag{13}$$

where the "correspondence" $Z_{d,d}$ of (3) is invariant under $\mathrm{Sl}(2,\mathbb{Z})\times\mathrm{SO}(d,d,\mathbb{Z})$.

The exact R^4 couplings. Our goal in this paper is to generalize these considerations to the eleven-dimensional membrane, and derive the exact non-perturbative R^4 couplings for M-theory compactified on a torus $\mathbb{T}^{d+1}$. Using arguments from supersymmetry and U-duality, it was indeed shown in [3, 4, 9] that the complete non-perturbative R^4 coupling could be written in terms of Eisenstein series of the U-duality group $E_{d+1(d+1)}$ [6]

$$f_{R^4} = \mathcal{E}^{E_{d+1(d+1)}(\mathbb{Z})}_{\mathbf{string};\,s=3/2} + \mathcal{E}^{E_{d+1(d+1)}(\mathbb{Z})}_{\mathbf{membrane};\,s=1}\,. \tag{14}$$

The Eisenstein series appearing in the r.h.s.[3] are eigenmodes of the U-duality invariant laplacian [4, 6], as required by supersymmetry [9, 10, 11]. The tree-level, one-loop and D-instanton terms in (1) arise upon expanding the Eisenstein series at weak coupling. For our present purposes, it is however more appropriate to consider the expansion in the limit of large volume, which commutes with the eleven-dimensional Lorentz group. Using the techniques in [6], we obtain

$$\begin{aligned} f_{R^4} = \frac{\pi^2 l_M^6}{3} &+ \sum_{m^i\in\mathbb{Z}^{d+1}} \frac{l_M^9}{[(m^i)^2]^{3/2}} + \pi \sum_{m^3\neq 0} \frac{l_M^9}{\sqrt{(m^3)^2}} + \\ &+ \pi l_M^6 \sum_{m^3\neq 0} \left[\frac{l_M^6}{(m^3)^2}\right]^{1/2} \mu(m^3) \exp\left(-\frac{2\pi}{l_M^3}\sqrt{(m^3)^2} + 2\pi i m^3 C_3\right) \times \\ &\times \left(1 + \mathcal{O}\left(\frac{1}{l_M^3}\right)\right) \end{aligned} \tag{15}$$

[3]The two terms correspond to different kinematic structures $(t_8 t_8 \pm \epsilon_8\epsilon_8)R^4$, and become equal for $d>2$ [4]. We work in units of Planck length, so that $V_{d+1}/l_M^9 = 1$.

which reveals a sum of "perturbative terms" and membrane instantons [12, 13], wrapping subtori $\mathbb{T}^3 \subset \mathbb{T}^{d+1}$ with wrapping number $m^{ijk} := m^3$ (similarly, C_3 denotes the 3-form gauge field C_{ijk}.) The instanton summation following from the previous computation is the number theoretic function

$$\mu\left(m^{ijk}\right) = \sum_{n|m^{ijk}} n\,. \tag{16}$$

It is therefore uniquely fixed by U-duality invariance.

Although the four-graviton amplitude is known exactly, the derivation we have outlined is very indirect, and tells little about the underlying fundamental degrees of freedom of M-theory. By contrast, a first-principle derivation, would give support to the fundamental nature of the purported degrees of freedom, as well as important practical insights into their quantization. In this paper, we test the proposal of the eleven-dimensional membrane as the elementary excitation of M-theory. In particular, we would like to rederive the summation measure (16) for membrane instantons, together with the perturbative terms in (15).

The BPS eleven-dimensional membrane. At this stage, given our lack of understanding of the fundamental degrees of freedom of M-theory, we need an act of faith. In this paper, we will contend that at least for the purposes of deriving this BPS amplitude,

(i) *the eleven-dimensional membrane provides the relevant degrees of freedom.* The membrane degrees of freedom are certainly needed, since the large volume expansion of the second term (and the first term for $d+1>3$) in (14) exhibit a sum over membrane instantons. For $d+1 \geq 6$, there are also 5-brane instantons, and therefore our computation will certainly not give the complete result there. Finally, the first term (for $d+1<3$) was reproduced in [14, 15] from a supergravity computation, and the second term for $d=1$ in [16], but in both cases the divergence had to be fixed by hand. One may hope that a hypothetical modular invariance of the membrane might render the amplitude manifestly finite;

(ii) *the only topology contributing is the torus* $\mathbb{T}^3$. This assumption is based on the fact that the only membrane instantons appearing in the expansion (15) are subtori $\mathbb{T}^3 \subset \mathbb{T}^d$. In particular, the amplitude should have $\mathrm{Sl}(3,\mathbb{Z})$ modular invariance on the membrane world-volume;

(iii) *all quantum fluctuations cancel, leaving only the contribution from the zero-modes.* This is in analogy with the string theory computation.

Points (ii) and (iii) are supported by a light-cone treatment of the four-graviton amplitude using the membrane vertex operators of [17]. In a hamiltonian formalism the

$\mathbb{T}^3 = S^1 \times \mathbb{T}^2$ topology corresponds to a one-loop closed supermembrane amplitude given by [18]

$$\mathcal{A}_4 = \mathrm{STr}\left(V^1\,\Delta\,V^2\,\Delta\,V^3\,\Delta\,V^4\,\Delta\right). \tag{17}$$

Here the V^i denote the graviton vertex operators separated by the propagator $\Delta = \int_0^\infty dt\,\exp[-t\,H]$ and the supertrace STr is over the Hilbert space of H. As we are dealing with a compactified theory the membrane spectrum becomes discrete in a semiclassical quantization of the non-zero winding sector [19]. By expanding around the classical winding configurations the hamiltonian splits into $H = H_{\rm class} + H_0 + H_{\rm int}$: the bosonic zero mode piece $H_{\rm class}$, a superharmonic oscillator of the quantum fluctuations H_0 with masses given by the windings, along with an interaction term $H_{\rm int}$. Therefore the Hilbert space is spanned by the discrete and continuous bosonic zero modes $|x^i\rangle$, the fermionic zero modes $|\mathcal{N}\rangle$ comprising the $\mathbf{44}+\mathbf{84}$ bosonic and $\mathbf{128}$ fermionic transverse states of the massless $D = 11$ supermultiplet, along with the discrete eigenstates $\|m\rangle\rangle$ of $H_0 + H_{\rm int}$. As the fermion zero modes θ_0 do not appear in the hamiltonian and sixteen insertions of θ_0 in the trace over $|\mathcal{N}\rangle$ are needed for a non-vanishing result, only the terms in V^i of highest fermionic degree (being four) enter the amplitude. The trace in (17) then factorizes as

$$\begin{aligned}\mathcal{A}_4 = \int_0^\infty dt \int d^{11}x\,\langle x|\;e^{-t\,H_{\rm class}}\,|x\rangle \times \sum_{\mathcal{N}} \langle\mathcal{N}|(-)^F\,V^1|_{\theta_0{}^4}\,V^2|_{\theta_0{}^4}\,V^3|_{\theta_0{}^4}\,V^4|_{\theta_0{}^4}\,|\mathcal{N}\rangle \times \\ \times \sum_m \langle\langle m\|(-)^F\;e^{-t\,(H_0+H_{\rm int})}\,\|m\rangle\rangle\,. \end{aligned}\tag{18}$$

The trace over the fermionic zero modes yields the appropriate tensor structure of the R^4 term [20, 21]

$$\begin{aligned}\mathrm{STr}\left(V^1|_{\theta_0{}^4}\,V^2|_{\theta_0{}^4}\,V^3|_{\theta_0{}^4}\,V^4|_{\theta_0{}^4}\right)_{|\mathcal{N}\rangle} &= \varepsilon^{\alpha_1\cdots\alpha_{16}}\,\Gamma^{j_1j_2}_{\alpha_1\alpha_2}\cdots\Gamma^{j_{15}j_{16}}_{\alpha_{15}\alpha_{16}}\,R_{j_1j_2j_3j_4}\cdots R_{j_{13}j_{14}j_{15}j_{16}} \\ &= \mathcal{C}^{pqrs}\,\mathcal{C}_{pq}{}^{tu}\,\mathcal{C}_{rt}{}^{vw}\,\mathcal{C}_{suvw} - \\ &\quad - 4\,\mathcal{C}^{pqrs}\,\mathcal{C}_p{}^t{}_r{}^u\,\mathcal{C}_t{}^v{}_q{}^w\,\mathcal{C}_{uvsw} + \text{Ricci-terms}\,,\end{aligned}\tag{19}$$

where $\Gamma^j_{\alpha_1\alpha_2}$ denote the transverse SO(9) Dirac matrices and $\mathcal{C}^{pqrs}$ $(p, q, \ldots = 0, \ldots 10)$ is the covariant Weyl tensor.[4] Due to supersymmetry and the discreteness of the spectrum of $H_0 + H_{\rm int}$, the (non zero-mode) last term in (18) is nothing but the Witten index associated with this Hamiltonian. Closer inspection reveals that this index equals one for all configurations with non-degenerate winding. We are thus left to compute the *classical* partition function $\mathcal{A}_{\rm BPS}$, to which we now turn.

Polyakov action of the membrane. In order to keep the modular symmetry $\mathrm{Sl}(3,\mathbb{Z})$ of the toroidal membrane manifest, let us use the Polyakov action of the

[4]This R^4 coupling based on SO(9) Dirac matrices is indeed equal to the SO(8) based $t_8\,t_8\,R^4$ coupling [22] in the Weyl sector, they differ however by R_{pq} and R terms, a fact also noted in [21].

euclidean membrane as our starting point:

$$S = \int d^3\sigma \sqrt{\gamma} \left(g_{ij} \gamma^{ab} \partial_a X^i \partial_b X^j - 1 \right) + i\epsilon^{abc} C_{ijk} \partial_a X^i \partial_b X^j \partial_c X^k \,. \tag{20}$$

Note that the cosmological constant term $\int \sqrt{\gamma}$ is necessary in order to ensure the classical equivalence with the Nambu-Goto action on-shell, and therefore reproduce the correct weight for the membrane instantons. In particular, the determinant of the metric does not decouple as in the string theory case. In line with our discussion above, we restrict ourselves to classical embeddings $X^i = m^i_a \sigma^a$ and constant worldvolume metrics $\gamma_{ab} = u \hat{\gamma}_{ab}$ where $\hat{\gamma}_{ab}$ is the unit volume metric. We therefore propose to consider the integral

$$\mathcal{A}_{\rm BPS} = \int_0^\infty \frac{du}{u} u^\alpha \int_{\mathcal{F}_{\rm Sl(3,\mathbb{Z})}} d\hat{\gamma} \sum_{m^i_a \in \mathbb{Z}^{3d+3}} e^{-\pi u (m^i_a g_{ij} \hat{\gamma}^{ab} m^j_b) + \pi u^3 + 2\pi i C_{ijk} \epsilon^{abc} m^i_a m^j_b m^k_c} \tag{21}$$

as a candidate to reproduce the exact R^4 couplings in (14) — we shall refer to it as the *covariant* amplitude. Here $\mathcal{F}_{\rm Sl(3,\mathbb{Z})}$ denotes the fundamental domain of $\mathrm{Sl}(3,\mathbb{Z}) \backslash \mathrm{Sl}(3,\mathbb{R}) / \mathrm{SO}(3)$, and the integration measure $d\hat{\gamma}$ is the $\mathrm{Sl}(3,\mathbb{R})$ invariant measure.[5] The exponent α is left unspecified at this stage, and will be fixed to $\alpha = 6$ later by comparing with the light-cone amplitude. The restriction to the fundamental domain makes the integral over $\hat{\gamma}$ manifestly finite, but, in sharp contrast to the string theory case, the integral over the volume factor u remains potentially divergent and requires regularization. A natural prescription is to rotate the integration domain of u to the semi-infinite line $u \in i\mathbb{R}^+$ in the complex plane. As in the integral representation of the Airy function $\mathrm{A}i(x) = \int_0^\infty du \cos(ux + u^3/3)$, the oscillations render the $u \to \infty$ integration finite, and yield the correct Nambu-Goto euclidean saddle points. The behavior in the small volume limit $u \to 0$ is however not controlled, and is at the source of the difficulties in membrane theory.

For explicit computations, it is useful to parameterize the metric through a 3-bein ν^a_c such that $\hat{\gamma}^{ab} = \nu^a_c \nu^b_c$,

$$\nu^a_b = \frac{1}{\lambda^{1/3}} \begin{pmatrix} \frac{1}{\sqrt{\tau_2}} & & \\ & \sqrt{\tau_2} & \\ & & \lambda \end{pmatrix} \begin{pmatrix} 1 & \tau_1 & \omega_1 \\ & 1 & \omega_2 \\ & & 1 \end{pmatrix}, \qquad X^i = m^i \sigma_1 + n^i \sigma_2 + p^i \sigma_3 \,. \tag{22}$$

For such a configuration, the classical action reads

$$S = \frac{\pi u}{\lambda^{2/3}} \left[\frac{|m^i + \tau n^i + \omega p^i|^2}{\tau_2} + \lambda^2 \left(p^i\right)^2 \right] - \pi u^3 + 2\pi i C_{ijk} m^i n^j p^k \,, \tag{23}$$

[5] $\mathrm{Sl}(3,\mathbb{Z})$ modular forms have been constructed in [4], with a different motivation. A mathematical introduction to $\mathrm{Sl}(n,\mathbb{Z})$ modular forms can be found in [23].

where $\tau = \tau_1 + i\tau_2$ and $\omega = \omega_1 + i\tau_2\,\omega_2$, and is integrated against the summation measure

$$\mathcal{A}_{\rm BPS} = \int_0^\infty \frac{du}{u}\, u^\alpha \int_{\mathcal{F}_{\rm Sl(3,\mathbb{Z})}} \frac{d\lambda\ d\tau_2\ d\tau_1\ d\omega_1\ d\omega_2}{\lambda^3 \tau_2^2} \sum_{(m^i,n^i,p^i)\in\mathbb{Z}^{3d+3}} e^{-S}\,. \tag{24}$$

The precise definition of the fundamental domain $\mathcal{F}_{\rm Sl(3,\mathbb{Z})}$ will be given below. At this stage we simply note that the Borel subgroup of $\mathrm{Sl}(3,\mathbb{Z})$ allows to restrict the integration of $(\tau_1, \omega_1, \omega_2)$ to a period $[-1/2, 1/2]$, while the integration on the volume factor u is not restricted by modular invariance.

From Polyakov to light-cone. In analogy with the string case, let us Poisson resum the windings m^i into momenta m_i. Changing variables from (u, τ_2, λ) to (x, y, t) with

$$u = (xyt)^{1/3}\,, \qquad \tau_2 = x^{1/2} t\,, \qquad \lambda = \frac{(yt)^{1/2}}{x^{1/4}} \tag{25}$$

the action becomes

$$\begin{aligned}\tilde{S} = \pi t \left[\left(m_i + C_{ijk} n^j p^k\right)^2 + x\left(n^i + \omega_2 p^i\right)^2 + y\left(p^i\right)^2 - xy \right] + \\ + 2\pi i m_i \left(\tau_1 n^i + \omega_1 p^i\right)\,,\end{aligned} \tag{26}$$

to be integrated against the measure

$$\int_{\mathcal{F}_{\rm Sl(3,\mathbb{Z})}\times\mathbb{R}^+} \frac{(xyt)^{\alpha/3} dx\ dy\ dt\ d\tau_1\ d\omega_1\ d\omega_2}{xy^2t^3} \sum_{m_i,n^i,p^i} e^{-\tilde{S}}\,. \tag{27}$$

Now the integral over (ω_2, x, y) is dominated by a saddle point at

$$\omega_2 = -\frac{n^i p^i}{(p^i)^2}\,, \qquad x = \left(p^i\right)^2, \qquad y = \left(n^i - \frac{n^j p^j}{(p^j)^2} p^i\right)^2. \tag{28}$$

The saddle point action reads

$$\tilde{S} = \pi t \left[\left(m_i + C_{ijk} n^j p^k\right)^2 + \left(n^i\right)^2 \left(p^i\right)^2 - \left(n^i p^i\right)^2 \right] + 2\pi i m_i \left(\tau_1 n^i + \omega_1 p^i\right)\,, \tag{29}$$

and dominates the amplitude

$$\mathcal{A}_{\rm BPS} = \int dt\ d\tau_1\ d\omega_1\ t^{(d-8)/2+\alpha/3} \sum_{m_i, m^{ij}} \left[(m^{ij})^2\right]^{(\alpha-6)/3} e^{-\tilde{S}}\,, \tag{30}$$

where $m^{ij} = n^i p^j - n^j p^i = \{X^i, X^j\}$, the Poisson bracket on the $\mathbb{T}^2$ membrane worldvolume. The real part of eq. (29) is the membrane light-cone hamiltonian

$$H = \frac{1}{P_+}\left[(P_i)^2 + \left\{X^i, X^j\right\}^2 \right], \tag{31}$$

where $1/P_+ = t$ is the Schwinger proper time. In particular, we recognize the U-duality invariant mass spectrum

$$\mathcal{M}^2 = \left(m_i + C_{ijk} m^{jk}\right)^2 + \left(m^{ij}\right)^2 . \tag{32}$$

The parameters τ_1 and ω_1 are Lagrange multipliers enforcing the constraints

$$m_i n^i = 0\,, \qquad m_i p^i = 0 \qquad \Longrightarrow \qquad m_i m^{ij} = 0 \tag{33}$$

which are precisely the BPS constraints. Finally, we fix the parameter α by matching the measure factor to the one-loop supergravity amplitude [14]. The latter receives contributions from four vertex insertions and an integral over $(10-d)$ continuous momenta, i.e. $\int \frac{dt}{t} t^4\, t^{(d-10)/2}$, yielding $\alpha = 6$. Happily, the factor $[(m^{ij})]^{(\alpha-6)/3}$ in (30) drops at the same time. Note that this derivation of the standard light-cone membrane hamiltonian from the Polyakov action is essentially a zero-mode version of the original argument in [24].

Fundamental domain of $\mathrm{Sl}(3,\mathbb{Z})$**.** In order to further analyze the proposed one-loop membrane amplitude (24), it is necessary to specify the fundamental domain of integration. The latter can be determined as follows. The modular group acts by right multiplication of the 3-bein (22) by $\mathrm{Sl}(3,\mathbb{Z})$ matrices. It is generated by its Borel subgroup, namely the discrete Heisenberg group

$$\begin{cases} \tau_1 \longrightarrow \tau_1 + a \\ \omega_1 \longrightarrow \omega_1 + b + c\tau_1\,, \qquad (a,b,c) \in \mathbb{Z}^3 \\ \omega_2 \longrightarrow \omega_2 + c \end{cases} \tag{34}$$

and its Weyl group, namely the group of permutations of the columns of the 3-bein. Using the Weyl group, we may thus choose a fundamental Weyl chamber, i.e. to order the squared norms of the three columns of (22) as

$$\frac{1}{\tau_2} < \frac{\tau_1^2 + \tau_2^2}{\tau_2} < \frac{\omega_1^2 + \tau_2^2 \omega_2^2}{\tau_2} + \lambda^2 \tag{35}$$

and restrict the Borel moduli to

$$-\frac{1}{2} < (\tau_1, \omega_1, \omega_2) < \frac{1}{2} . \tag{36}$$

In other words, $\tau = \tau_1 + i\tau_2$ is restricted to the standard fundamental domain of $\mathrm{Sl}(2,\mathbb{Z})$, $\omega_1 + \tau\omega_2$ is restricted to the torus of complex modulus τ, while the extra modulus λ takes values on the half-line

$$\lambda^2 \tau_2 > \tau_1^2 + \tau_2^2 (1-\omega_2)^2 - \omega_1^2 . \tag{37}$$

The volume modulus u, on the other hand, should be integrated from 0 to ∞.

Membrane instantons. Let us now consider the large volume expansion of (24), or equivalently the Fourier expansion in the periodic modulus C_{ijk}. In analogy with the one-loop string amplitude, one should decompose the summation over the integers (m^i, n^i, p^i) into their orbits under $\mathrm{Sl}(3,\mathbb{Z})$, and unfold the integration region accordingly. Orbits of $\mathrm{Sl}(3,\mathbb{Z})$ are classified by the rank of the matrix (m^i, n^i, p^i), hence we have now four different contributions. The simplest case to consider is the non-degenerate orbit of rank 3: the fundamental domain can be unfolded to the full "upper half plane"

$$(\tau_2, \lambda, u) \in \mathbb{R}^+ \,, \qquad (\tau_1, \omega_1, \omega_2) \in \mathbb{R} \,. \tag{38}$$

The integral over the Borel parameters $(\tau_1, \omega_1, \omega_2)$ of (23) and (24) is exactly gaussian, and gives a classical action

$$\Re(S) = \frac{\pi u}{\lambda^{2/3}\tau_2 (m^{ij})^2 \left(p^i\right)^2} \Big\{ \left(m^{ijk}\right)^2 \left(p^i\right)^2 + \tau_2 \lambda^2 \left(m^{ij}\right)^2 \left[(p^i)^2\right]^2 + \\ + \tau_2^2 \left[(m^{ij})^2\right]^2 \Big\} - \pi u^3 \tag{39}$$

supplemented by the same coupling $2\pi i C_{ijk} m^i n^j p^k$ to the background 3-form. Here $m^{ij} = n^{[i} p^{j]}$ and $m^{ijk} = m^{[i} n^j p^{k]}$. In terms of the (x, y, t) variables this is

$$S = \pi \left\{ \frac{(m^{ijk})^2}{t (m^{ij})^2} + \frac{(m^{ij})^2}{(p^i)^2} xt + \left(p^i\right)^2 yt \right\} - \pi xyt + 2\pi i C_{ijk} m^{ijk} \,. \tag{40}$$

Finally, we can integrate over (x, y, t), and obtain in the saddle point approximation

$$A_{\text{nondeg}} = \sum_{m^{ijk}} \mu' \left(m^{ijk}\right) \frac{e^{-2\pi\sqrt{(m^{ijk})^2} + 2\pi i C_{ijk} m^{ijk}}}{\sqrt{(m^{ijk})^2}} \,. \tag{41}$$

The summation measure m^{ijk} can be computed by noting that the sum has to go over $\mathrm{Sl}(3,\mathbb{Z})$ orbits of the integer matrix (m^i, n^i, p^i): a representative can be chosen as

$$\begin{pmatrix} m^i \\ n^i \\ p^i \end{pmatrix} = \begin{pmatrix} m & j & k & m^4 & m^5 & \cdots \\ 0 & n & l & n^4 & n^5 & \cdots \\ 0 & 0 & p & p^4 & p^5 & \cdots \end{pmatrix} \tag{42}$$

with $m, n, p > 0$ and $0 \le j < n, 0 \le k, l < p$. Hence the summation measure is given by the number-theoretic function

$$\mu' \left(m^{ijk}\right) = \sum_{n | m^{ijk}} \sum_{p | (m^{ijk}/n)} n p^2 \,. \tag{43}$$

Importantly, this geometric summation measure disagrees with the summation measure predicted by U-duality in (24). In addition to these instanton contributions, the amplitude also contains perturbative terms corresponding to the degenerate orbits of rank 0,1,2, which should reproduce the first three terms in (15), respectively.

In view of the discrepancy already observed at the level of the non-degenerate orbit, we will refrain from discussing those in detail. We however observe that, had our semi-classical proposal succeeded in reproducing the membrane instantons, it would also have produced the correct perturbative terms, since those are related by U-duality.

Intermezzo. Before proceeding, some comments are in order.

(i) the covariant amplitude (21) reproduces both the correct BPS mass spectrum (32) and classical saddle point values (41) for membrane instantons. This comes as no surprise, given the equivalence between the Polyakov, light-cone and Nambu-Goto formulations of the membrane at the classical level.

(ii) one advantage of the covariant amplitude is its manifest modular invariance under $\mathrm{Sl}(3,\mathbb{Z})$, which is hidden in the standard light-cone approach. One should therefore restrict to the fundamental domain of $\mathrm{Sl}(3,\mathbb{Z})$ in order to avoid multiple countings; in contrast to the string theory case, modular invariance however does not imply the finiteness of membrane theory yet, since the integration on the volume u remains unbounded.

(iii) while the classical saddle points are correctly reproduced, the covariant amplitude predicts the wrong instanton summation measure; in particular, it is not U-duality invariant — in fact, this is easily seen in the $d=2$ case with $C=0$. Although the classical Bianchi identities and equations of motion following from the Polyakov action for the membrane do fall into U-duality multiplets [25], this is not sufficient to ensure the invariance of the classical partition function. In particular, U-duality requires the exchange of momentum P_i and the composite winding number $\{X^i, X^j\}$, under which the measure is not manifestly invariant.

(iv) in order to generate the proper R^4 amplitude, the membrane partition function should be an eigenmode of a combination of the $\mathrm{Sl}(3,\mathbb{R})/\mathrm{SO}(3)$ and $E_{d+1(d+1)}$ laplacian. This is not the case, even though the values of the saddle points are indeed eigenmodes of the $E_{d+1(d+1)}$ laplacian.

(v) the discrepancy between the geometric summation measure (43) and the one predicted by U-duality in (24) implies that the contribution of the non-zero modes to the covariant amplitude is not unity as a naive extrapolation from the string case might have suggested, but instead should be given by the ratio $\mu(N)/\mu'(N)$ where $N = m^{ijk}$. This is an interesting prediction about the interacting world-volume theory of the membrane that would be interesting to verify.

Keeping with our act of faith, we now propose that the shortcomings of our proposal (21) can be repaired by enforcing modular invariance and U-duality explicitly. More precisely, our task is *to construct a partition function* $\Xi_{d+1}(\gamma_{ab}; g_{ij}, C_{ijk})$ *for mappings of* $\mathbb{T}^3$ *into* $\mathbb{T}^{d+1}$*, invariant under* $\mathrm{Sl}(3,\mathbb{Z})\times E_{d+1(d+1)}(\mathbb{Z})$, such that, upon integration on the fundamental domain of $\mathrm{Sl}(3,\mathbb{Z})$,[6] we reproduce the non-perturbative R^4 couplings:

$$\int_{\mathcal{F}(\mathrm{Sl}(3,\mathbb{Z}))} \Xi_{d+1} = \mathcal{E}^{E_{d+1(d+1)}(\mathbb{Z})}_{\mathbf{string};\, s=3/2} + \mathcal{E}^{E_{d+1(d+1)}(\mathbb{Z})}_{\mathbf{membrane};\, s=1}\,. \tag{44}$$

More generally, integrating Ξ_{d+1} against an $\mathrm{Sl}(3,\mathbb{Z})$ automorphic form should produce a lift to a $E_{d+1(d+1)}(\mathbb{Z})$ modular form. This is a standard problem in the theory of automorphic forms, that goes under the name of "theta" correspondence.

Theta correspondences. In order to gain further insight into this problem, it is useful to return to the one-loop string amplitude (2) once again. The invariance under $\mathrm{Sl}(2,\mathbb{Z})\times \mathrm{SO}(d,d,\mathbb{Z})$ of the partition function $Z_{d,d}(\tau; g, B)$ of the string zero-modes is a rather subtle phenomenon: only the modular invariance $\mathrm{Sl}(2,\mathbb{Z})$ is manifest in the lagrangian picture (3), while the $\mathrm{SO}(d,d,\mathbb{Z})$ symmetry becomes apparent in the hamiltonian picture (6), at the cost of losing the manifest modular symmetry. Indeed, the partition function (3) is a special case of a theta series

$$\theta_{\mathrm{Sp}(g)}(\Omega_{AB}) = \sum_{m^A\in\mathbb{Z}^g} e^{-\pi m^A \Omega_{AB} m^B} \tag{45}$$

invariant under the symplectic group $\mathrm{Sp}(g,\mathbb{Z})$ acting by fractional linear transformations on the period matrix $\Omega \in \mathrm{Sp}(g,\mathbb{R})/\,\mathrm{U}(g)$, through Poisson resummation on the integers m^A. In fact,

$$Z_{d,d}(\tau; g, B) = \theta_{\mathrm{Sp}(2d)}(\tau\otimes(g+b))\,, \tag{46}$$

where the tensor product provides an embedding $\mathrm{Sl}(2)\times\mathrm{SO}(d,d)\subset\mathrm{Sp}(2d)$. The pair $\mathrm{Sl}(2)\times\mathrm{SO}(d,d)$, such that the centralizer of either group is equal to the other, is known as a dual pair [27]. In fact, there exists a discrete set of elements in $\mathrm{Sp}(2d)$ preserving the decomposition $\mathrm{Sl}(2)\times\mathrm{SO}(d,d)\subset\mathrm{Sp}(2d)$, so that the partition function $Z_{d,d}$ is actually invariant under a bigger group mixing target-space and world-sheet [28]. If our proposal holds, we would therefore also have a symmetry mixing target-space and world-volume in membrane theory.

Our task is therefore to find a generalization of $\theta_{\mathrm{Sp}(g)}$, such that the restriction to an $\mathrm{Sl}(3,\mathbb{Z})\times E_{d+1(d+1)}(\mathbb{Z})$ subgroup will provide our Ξ_{d+1}. Remarkably, such problems are the subject of much interest in the theory of automorphic forms, and many results are available in the mathematical literature. The relevant results for us are the following:

[6] Here we include the integration on the volume factor $\det(\gamma)$ in Ξ_{d+1}.

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G	n	correspondences
E_8	29	$E_7 \times \mathrm{Sl}(2),\ E_6 \times \mathrm{Sl}(3),\ \mathrm{SO}(5,5) \times \mathrm{Sl}(4),\ \mathrm{Sl}(3) \times \mathrm{Sl}(2),\ F_4 \times G_2$
E_7	17	$\mathrm{SO}(6,6) \times \mathrm{Sl}(2),\ \mathrm{Sl}(6) \times \mathrm{Sl}(3),\ G_2 \times \mathrm{Sl}(2),\ G_2 \times \mathrm{Sp}(6),\ F_4 \times \mathrm{Sl}(2)$
E_6	11	$\mathrm{Sl}(3) \times \mathrm{Sl}(3) \times \mathrm{Sl}(3),\ \mathrm{Sl}(6) \times \mathrm{Sl}(2),\ G_2 \times \mathrm{Sl}(3)$
$\mathrm{SO}(5,5)$	7	$\mathrm{Sl}(2) \times \mathrm{Sl}(2) \times \mathrm{Sl}(4),\ \mathrm{Sp}(4) \times \mathrm{Sp}(4)$
$\mathrm{Sl}(8)$	7	$\mathbb{R}^+ \times \mathrm{Sl}(3) \times \mathrm{Sl}(5)$
$\mathrm{Sl}(5)$	4	$\mathbb{R}^+ \times \mathrm{Sl}(2) \times \mathrm{Sl}(3)$
F_4	8	$G_2 \times \mathrm{Sl}(2),\ \mathrm{Sl}(3) \times \mathrm{Sl}(3),\ \mathrm{Sp}(6) \times \mathrm{Sl}(2)$
G_2	3	$\mathrm{Sl}(2) \times \mathrm{Sl}(2)$

Table 1: Non-exhaustive list of correspondences for exceptional groups, and dimension n of the "smallest" representation. All groups are assumed in the split real form.

(i) A classification of dual pairs $G_1 \times G_2 \subset G$ exists [29, 30]. The part of it relevant for our purposes is reproduced in table 1. For the problem at hand, we are interested in the following pairs,

$$\begin{aligned} \mathbb{R}^+ \times \mathrm{Sl}(3) \times \mathrm{Sl}(2) &\subset \mathrm{Sl}(5)\,, \\ \mathbb{R}^+ \times \mathrm{Sl}(3) \times \mathrm{Sl}(2) \times \mathrm{Sl}(3) &\subset \mathrm{Sl}(3)^3 \subset E_{6(6)}\,. \\ \mathbb{R}^+ \times \mathrm{Sl}(3) \times \mathrm{Sl}(5) &\subset \mathrm{Sl}(8)\,, \\ \mathbb{R}^+ \times \mathrm{Sl}(3) \times \mathrm{SO}(5,5) &\subset \mathrm{Sl}(4) \times \mathrm{SO}(5,5) \subset E_{8(8)}\,, \\ \mathrm{Sl}(3) \times E_{6(6)} &\subset E_{8(8)} \end{aligned} \tag{47}$$

which allow to lift $\mathrm{Sl}(3,\mathbb{Z})$ modular forms to $E_{d+1(d+1)}(\mathbb{Z})$ for $d = 1,2,3,4,5$, respectively. We have included the $d = 5$ example for completeness, although for M-theory on $\mathbb{T}^6$, due to M5-brane instantons, one should not expect that the membrane provide all the relevant degrees of freedom. This last example is nevertheless quite attractive, since it contains all the other ones, and involves no volume factor at all: the integral becomes manifestly finite when restricting to the fundamental domain of $\mathrm{Sl}(3,\mathbb{Z})$. The other examples listed in table 1 might have some relevance in other M-theoretical contexts.

(ii) For any simply-laced split group G [31, 32, 33] (and for some non simply-laced groups, see e.g. [34]), there exists an essentially unique minimal unitary representation π of G, on a Hilbert space H of functions of n variables [35]. This is analogous to the Weyl representation of the symplectic group $\mathrm{Sp}(g)$ on functions of g variables [36], where the generators of a maximal Heisenberg subgroup are represented by the translations and multiplications by characters of the Schrödinger representation, and the action of the Heisenberg subgroup is extended to that of the full group G by a Weyl reflection represented by the Fourier transform with respect to the n variables. These minimal representations, and in particular the 29-dimensional representation of $E_{8(8)}$ and the associated modular forms, remain to be constructed explicitly. A first step in this direction was taken in [37] with the explicit construction of a quasi-conformal (non-unitary) representation of $E_{8(8)}$ on $\mathbb{R}^{57}$. The dimension of the minimal representation was first derived in [35], and is shown in table 1 for various groups of interest. In contrast to the symplectic case, the representation of the Heisenberg subalgebra requires some cubic characters, and the closure of the algebra implies some important identities for their Fourier transform.

(iii) There exists a vector $f \in H$ invariant under the maximal compact subgroup $K \subset G$; and a distribution δ in H^* invariant under a discrete subgroup $G(\mathbb{Z})$ generated by the integer translations and Weyl reflections. From these two objects one can construct an automorphic form θ on $G(\mathbb{Z})\backslash G/K$, given by the matrix element $\theta(\Omega) = \langle \delta, \pi(\Omega) f \rangle$. This generalizes the theta function for the symplectic case, where $f(x) = e^{-\pi(x^A)^2}$, and $\delta(x) = \sum_{m \in \mathbb{Z}^g} \delta(x-m)$. In general, one expects δ to involve a sum on a n-dimensional lattice, where n is the dimension in which the minimal representation is realized. The invariance of θ under Weyl reflection requires a Poisson resummation, which follows from the Fourier transform identity for the non-gaussian character found in (ii).

Unfortunately, although theta correspondences are known to exist, they remain elusive objects. It would therefore be very desirable to construct explicit formulas analogous to (3). From the mathematical point of view, this would provide a wealth of new Poisson resummation formula for non-gaussian characters[7] generalizing the ones found in [38], and may have interesting applications in number theory. For our purposes, it would be very interesting to compare them to our proposal in (24) in order to extract the quantum measure for the membrane theory. An exciting possibility is that the $\mathrm{Sl}(3) \times E_{6(6)} \subset E_{8(8)}$ exceptional theta correspondence in (47) would reproduce the full R^4 couplings: in addition to curing the small volume divergence of the membrane, it would also imply that BPS membranes contain five-branes. This we believe would be an important hint that membranes may indeed be the fundamental degrees of freedom of M-theory.

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[7] In fact cubic, since the membrane couples to the C-field through a cubic coupling, while the Polyakov action becomes also cubic when the volume u is eliminated.

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